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Minimizing Increasing Star-shaped Functions Based on Abstract Convexity ⋆

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Abstract. We study a broad class of increasing non-convex functions whose level sets are star shaped with respect to infinity. We show that these functions (we call them ISSI functions) are abstract convex with respect to the set of min-type functions and exploit this fact for their minimization. An algorithm is proposed for solving global optimization problems with an ISSI objective function and its numerical performance is discussed.

Key words: Abstract convexity, Global optimization, Increasing function

1. Introduction

There exist a number of techniques for finding a global optimum of a non-convex function, each of them has advantages and shortcomings. As there is no universal algorithm which would be efficient for any global optimization problem, it is most important to create conceptual and numerical schemes which will lead to the solution of problems with objective functions and/or constraints with special properties.

In this paper we consider an approach to global optimization which is based on the use of abstract convexity, in other terms, 'convexity without linearity' (see [13, 16, 23]). It allows to extend the convex analysis to broad classes of non-convex problems and to generalize known methods of cutting plane or bundle type for optimization problems with abstract convex objective functions.

There are a number of numerical algorithms which are actually based on abstract convexity ideas. For instance, the algorithm of Mladineo (see [15]) for maximizing Lipschitz functions can be considered as a method of abstract convex programming with quadratic convex functions as majorants of the objective. Recently, the cutting angle method for minimizing the so-called increasing convex-along-rays (ICAR) functions has been proposed which is based on the representation of the objective as the supremum of min-type functions (see [2]).

In this paper we consider a class of objective functions which are increasing and have star-shaped (with respect to ∞) level sets (we call them ISSI functions). Such

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functions form a lattice and have a number of nice properties simplifying their optimization. The mathematical programming problems with ISSI objective functions have various applications in practise, especially in mathematical economics which makes them very significant. For instance Intriligator [11] considers the functions with this property as the main kind of production functions.

In general, an ISSI function has multiple local extrema and it is neither convex nor concave. Hence it is impossible to apply to its minimization local search methods or methods of concave minimization directly. Since ISSI functions can be approximated by the so-called min-type functions due to their abstract convexity properties, the problem of minimizing an ISSI function can be reduced to a sequence of minimax problems, or to a sequence of problems of mixed integer linear programming. Certainly, each subproblem of this type is difficult to solve and its complexity grows fast with respect to the dimensionality of the initial problem. However, using special reduction techniques, it is possible to obtain good numerical performance, if the number of variables is not large. It can be advantageous to combine methods based on abstract convexity with a local search, thus obtaining hybrid methods. In this case the abstract convex programming is used in order to find an approximate solution and the local search improves it. It is important to note that many optimization problems can be reduced to a problem with an ISSI objective function by a suitable transformation of variables.

2. Preliminaries

In the sequel we shall need some definitions from abstract convexity (for detailed presentation see [16, 13, 23]).

DEFINITION 2.1. Let X be an arbitrary set and $H : X \to \mathbb{R}$ be a set of functions. A function $f : X \to \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ is called *abstract convex* with respect to H or H-convex if there is a set $U \subset H$ such that

$$f(x) = \sup\{h(x) : h \in U\} \quad \text{for all} \quad x \in X.$$

A function $f : X \to \mathbb{R}_{-\infty}$ is called *abstract concave* with respect to H or H-concave if there is a set $U \subset H$ such that

$$f(x) = \inf\{h(x) : h \in U\} \quad \text{for all} \quad x \in X.$$

The set H in this definition is called sometimes the *set of elementary functions*. Let

$$\mathrm{s}_-(f) = \{h \in H : h \leqslant f\}, \qquad \mathrm{s}_+(f) = \{h \in H : h \geqslant f\}.$$

It follows immediately from the definition that a function f is H-convex (respectively H-concave) if and only if $f(x) = \sup\{h(x) : h \in \mathrm{s}_-(f)\}$ (respectively $f(x) = \inf\{h(x) : h \in \mathrm{s}_+(f)\}$).

Applications of abstract convexity are mainly based on the notion of the subgradient. There are various approaches to the definition of subgradients (see [16, 23, 20]). For our purpose the most suitable is the following one.

DEFINITION 2.2. A function $h \in H$ is called a *subgradient* of an H-convex function f at a point x_o if $h(x) \leqslant f(x)$ for all $x \in X$ and $f(x_o) = h(x_o)$. A function $h \in H$ is called a *supergradient* of a H-concave function f at a point x_o if $h(x) \geqslant f(x)$ for all $x \in X$ and $h(x_o) = f(x_o)$.

The set of all subgradients (supergradients) of a H-convex (H-concave) function f at the point x_o is called the *subdifferential* (the *superdifferential*) of f at the point x. It follows directly from the definition that the subdifferential of H-convex function f at a point x_o is not empty if and only if $f(x_o) = \max\{h(x_o) : h \in \mathrm{s}_-(f)\}$.

3. Star-shaped functions

A real-valued function f defined on a cone K in a n-dimensional space $\mathbb{R}^n$ is called 0-star-shaped (or star-shaped with respect to 0) if $f(\lambda x) \leqslant \lambda f(x)$ for all $x \in K$ and all nonnegative $\lambda \leqslant 1$. This term was introduced in a slightly different situation by Pallaschke and Rolewicz ([16]). If f is 0-star-shaped then its lower level sets $\{x : f(x) \leqslant c\}$ are 0-star-shaped for all $c > 0$. Recall that a set A is called 0-star-shaped if

$$x \in A, \; \lambda \in [0, 1] \implies \lambda x \in A.$$

Of course there exist non 0-star-shaped functions with 0-star-shaped level sets. A full description of such functions is given in [22] (for the case, when the cone K coincides with the entire space).

The complement to a 0-star-shaped set is the so-called $+\infty$-star-shaped set (see [19] for details), that is a set B with the property

$$x \in B, \; \lambda \geqslant 1 \implies \lambda x \in B.$$

Convex $+\infty$-star-shaped sets have been studied by many authors (see for example [17, 4, 14] and references therein).

Having in mind this terminology, we introduce the following definition: A real-valued function f defined on a cone K is called $+\infty$-*star-shaped* if

$$f(\lambda x) \geqslant \lambda f(x) \quad \text{for all} \quad x \in K, \; \lambda \in [0, 1] \tag{1}$$

Note that $f(0) \geqslant 0$ for a $+\infty$-star-shaped function.

It is easy to check that f is $+\infty$-star-shaped if and only if

$$f(\mu x) \leqslant \mu f(x) \quad \text{for all} \quad x \in K, \; \mu \geqslant 1 \tag{2}$$

Indeed, let f be $+\infty$-star-shaped . Let $\mu \geqslant 1, \; x \in K$ and $\mu x = x'$. Since $x = \lambda x'$ with $\lambda = 1/\mu \leqslant 1$, it follows that $f(x) \geqslant \lambda f(x')$. We have proved that (1) implies

(2). The same reasoning shows that (2) implies (1). We will consider mainly $+\infty$-star-shaped functions defined on the cone $\mathbb{R}^n_+$ of all n-vectors with nonnegative coordinates. Sometimes we will consider the restriction of such functions on the cone $\mathbb{R}^n_{++}$ of all vectors with positive coordinates.

The cone $\mathbb{R}^n_+$ introduces the usual coordinate-wise order relation in the space $\mathbb{R}^n$. We will use the following notation: $x \geqslant y$ if $x - y \in \mathbb{R}^n_+$; $x > y$ if $x \geqslant y$ and $x \neq y$ and $x \gg y$ if $x - y \in \mathbb{R}^n_{++}$. A function f defined on the cone $\mathbb{R}^n_+$ is called increasing if $x \geqslant y$ implies $f(x) \geqslant f(y)$.

We will study in this paper increasing $+\infty$-star-shaped (briefly ISSI) functions. Since an ISSI function f is increasing and $f(0) \geqslant 0$ it follows that $f(x) \geqslant 0$ for all $x \in \mathbb{R}^n_+$.

Let us check that an ISSI function f is continuous on the cone $\mathbb{R}^n_{++}$. In fact let $x \gg 0$ and $x_k \to x$. Let $\varepsilon > 0$. Then for sufficiently large k we have $(1-\varepsilon)x \leqslant x_k \leqslant (1+\varepsilon)x$. Using properties of the function f and (1), (2) we have

$$(1-\varepsilon)f(x) \leqslant f((1-\varepsilon)x) \leqslant f(x_k) \leqslant f((1+\varepsilon)x) \leqslant (1+\varepsilon)f(x).$$

The continuity has been proved.

The set $\mathcal{F}(ISSI)$ of ISSI functions possesses the following useful properties.

(1) $\mathcal{F}(ISSI)$ is a convex cone: if f, g are ISSI and λ, μ are positive numbers then also $\lambda f + \mu g$ is ISSI;
(2) $\mathcal{F}(ISSI)$ is a conditionally complete lattice; more precisely, for an arbitrary family $(f_\alpha)_{\alpha \in A}$ of ISSI functions the point-wise infimum $\inf_{\alpha \in A} f_\alpha$ is ISSI as well. If the family $(f_\alpha)_{\alpha \in A}$ is bounded above, that is there exists a ISSI function f such that $f_\alpha \leqslant f$ for all $\alpha \in A$, then the point-wise supremum $\sup_{\alpha \in A} f_\alpha$ is also ISSI.
(3) If the family f_k of ISSI functions point-wise converges to a finite function f then f is also an ISSI function.

We need the following definitions. An increasing function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is called *convex-along-rays* (briefly ICAR(X)) if for each $x \in \mathbb{R}^n_+$ the function $f_x =: f(\alpha x)$ is convex on $(0, +\infty)$. An increasing function f is called *concave-along-rays* (briefly ICAR(E)) if the function $f_x(\alpha)$ is concave on $(0, +\infty)$ for each $x \in \mathbb{R}^n_+$. ICAR(X) functions are studied in details in [20]. It is easy to check that an ICAR(E) function f such that $f(0) \geqslant 0$ is ISSI. In fact we have for each $x \in \mathbb{R}^n_+$ and $\lambda \in [0, 1]$:

$$f(\lambda x) = f_x(\lambda) = f_x(\lambda + (1-\lambda)0) \geqslant \lambda f_x(1) + (1-\lambda)f(0) \geqslant \lambda f(x).$$

Let us give some examples of ISSI functions.

EXAMPLE 3.1. An increasing positively homogeneous of degree $0 < \delta \leqslant 1$ function f is ISSI. Indeed such a function is ICAR(E). (If $\delta \geqslant 1$ then f is an

ICAR(X) function). In particular a Cobb-Douglas function

$$f(x) = Cx_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{with} \quad \alpha_i \geqslant 0, \sum_i \alpha_i \leqslant 1 \tag{3}$$

is ISSI. Note that if $\sum_i \alpha_i \geqslant 1$ then the Cobb-Douglas function is ICAR(X). The function $f_p(x) = (\sum_i x_i^p)^{\frac{1}{p}}$ $(x \in \mathbb{R}^n_+)$ with $p > 0$ is also ISSI. Note that this function is convex if $p \geqslant 1$ and concave if $p \leqslant 1$.

EXAMPLE 3.2. A concave increasing function f defined on $\mathbb{R}^n_+$ with $f(0) \geqslant 0$ is ISSI. Indeed f is ICAR(E). In particular the sum of two functions of the form (3) is concave increasing.

EXAMPLE 3.3. The point-wise supremum f of a family of ICAR(E) functions $(f_\alpha)_{\alpha \in A}$ is an ISSI function, provided $f(x) < +\infty$ for all x. Note that this function is not necessary ICAR(E).

EXAMPLE 3.4. Let $f_1, \ldots, f_k$ be increasing positively homogeneous of degree $0 < \delta_k \leqslant 1$ functions. Then their sum, minimum and maximum are ISSI functions.

4. ISSI functions and IPH functions

There exist close relations between ISSI functions and the so-called IPH functions (IPH stands for *increasing* and *positively homogeneous of degree one* functions).

In order to establish these relations we need the following definition: Let f be a function defined on a cone $\mathbb{R}^n_+$. A function $\hat{f}$ defined on the cone

$$\mathbb{R}^{n+1}_* = \{(x, \lambda) : x \in \mathbb{R}^n_+, \ \lambda > 0\} \cup \{0, 0\} \tag{4}$$

by the formula

$$\hat{f}(x, \lambda) = \lambda f(\frac{x}{\lambda}), \quad (x \in K, \ \lambda > 0), \quad \hat{f}(0, 0) = 0. \tag{5}$$

is called the *positively homogeneous extension* of the function f.

The following result has been proved in [1]. We give its proof for completeness.

THEOREM 4.1. *A function f defined on $\mathbb{R}^n_+$ is ISSI if and only if its positively homogeneous extension $\hat{f}(x, \lambda)$ is increasing in both variables x, λ.*

Proof: Let f be an ISSI function. Since $f(x) \geqslant 0$ for all $x \in \mathbb{R}^n_+$ it follows that $\hat{f}(x, \lambda) = \lambda f(x/\lambda) \geqslant 0$. Consider now two points (x_1, λ_1) and (x_2, λ_2) with $x_1, x_2 \in \mathbb{R}^n_+, \ x_1 \geqslant x_2$ and $\lambda_1 \geqslant \lambda_2 > 0$. We have

$$\hat{f}(x_1, \lambda_1) = \lambda_1 f(\frac{x_1}{\lambda_1}) \geqslant \lambda_1 f(\frac{x_2}{\lambda_1}) = \lambda_1 f(\frac{\lambda_2}{\lambda_1}\frac{x_2}{\lambda_2}) \geqslant \lambda_2 f(\frac{x_2}{\lambda_2}) = \hat{f}(x_2, \lambda_2).$$

Thus $\hat{f}$ is an increasing function. We now assume that $\hat{f}$ is increasing. Then $\hat{f}(x,\lambda) \geqslant \hat{f}(0,0) = 0$, in particular $f(0) = \hat{f}(0,1) \geqslant 0$. If $x_1 \geqslant x_2$ then $f(x_1) = \hat{f}(x_1,1) \geqslant \hat{f}(x_2,1) = f(x_2)$. Thus f is increasing. Let $\lambda \in (0,1)$. Then $(\lambda x, \lambda) \leqslant (\lambda x, 1)$, hence

$$\lambda f(x) = \hat{f}(\lambda x, \lambda) \leqslant \hat{f}(\lambda x, 1) = f(\lambda x).$$

If $\lambda = 0$ then $0 = \lambda f(x) \leqslant f(0) = f(\lambda x)$. Thus f is ISSI function. □

REMARK 4.1. The same reasoning shows that the following assertion holds: a function f defined on $\mathbb{R}^n_+$ is 0-star-shaped and decreasing if and only if the positively homogeneous extension $\hat{f}$ is decreasing in both variables.

Theorem 4.1 allows to study ISSI functions with the help of IPH functions, which are simpler.

5. IPH functions

We study increasing positively homogeneous of the first degree(IPH) functions in this section. The simplest examples of IPH functions are given by min-type and max-type functions:

$$l_-(x) = \min_{i \in \mathcal{T}(l)} l_i x_i := \langle l, x\rangle^- \tag{6}$$

and

$$l_+(x) = \max_{i \in I} l_i x_i := \langle l, x\rangle^+. \tag{7}$$

Here $l = (l_1, \dots, l_n) \in \mathbb{R}^n_+$, $I = \{1, 2, \dots, n\}$ and $\mathcal{T}(l) = \{i \in I : l_i > 0\}$. It is assumed the the minimum over the empty set is equal to zero.

We denote $\min_{i \in \mathcal{T}(l)} l_i x_i$ by $\langle l, x\rangle^-$ and $\max_{i \in I} l_i x_i$ by $\langle l, x\rangle^+$.

We will show in this section that each IPH function defined on $\mathbb{R}^n_+$ is abstract convex with respect to the set $\mathcal{L}^-$ of min-type functions of the form (6) and the restriction of each IPH function on the cone $\mathbb{R}^n_{++}$ is abstract concave with respect to the set $\mathcal{L}^+$ of all max-type functions of the form (7).

For this purpose we will consider the subdifferential $\partial p^-(y)$ of an IPH function p with respect to $\mathcal{L}^-$ and the superdifferential $\partial p^+(y)$ of the function p with respect to $\mathcal{L}^+$ (see Definition 2.2). By this definition

$$\partial^- p(y) = \{l \in \mathbb{R}^n_+ : \langle l, x\rangle^- \leqslant p(x)\ \forall x \in \mathbb{R}^n_+,\ \langle l, y\rangle^- = p(y)\}.$$

$$\partial^+ p(y) = \{l \in \mathbb{R}^n_+ : \langle l, x\rangle^+ \geqslant p(x)\ \forall x \in \mathbb{R}^n_+,\ \langle l, y\rangle^+ = p(y)\}.$$

For $l \in \mathbb{R}^n_+$ we define the vector $l^{-1} = \dfrac{1}{l}$ by the formula:

$$\frac{1}{l} = \begin{cases} \dfrac{1}{l_i} & i \in \mathcal{T}(l) \\ 0 & i \notin \mathcal{T}(l) \end{cases}$$

Clearly, $\langle l, \frac{1}{l} \rangle^- = \langle l, \frac{1}{l} \rangle^+ = 1$ for $l \neq 0$.

PROPOSITION 5.1. *Let p be an IPH function. Then the set $\partial p^-(y)$ is not empty for all $y \geqslant 0$; if $p(y) > 0$ then*

$$\partial^- p(y) = \{l : \mathcal{T}(l) \subset \mathcal{T}(y) : l \geqslant \frac{p(y)}{y_l},\ p(\frac{1}{l}) = 1\} \tag{8}$$

where the coordinate $(y_l)_i$ of the vector y_l is equal to coordinate y_i of the vector y if $i \in \mathcal{T}(l)$ and equal to zero if $i \notin \mathcal{T}(l)$.

Proof: If $p(y) = 0$ then $0 \in \partial^- p(y)$ and therefore $\partial^- p(y)$ is nonempty. Consider now a point y with $p(y) > 0$. Let

$$A = \{l : \mathcal{T}(l) \subset \mathcal{T}(y),\ l \geqslant \frac{p(y)}{y_l},\ p(\frac{1}{l}) \geqslant 1\}.$$

It is easy to check that A is equal to the set on the right hand side of (8). Indeed let $l \in A$. The relations $\mathcal{T}(l) \subset \mathcal{T}(y)$ and $l \geqslant p(y)/y_l$ show that $y_l \geqslant p(y)/l$. Since p is IPH, $y \geqslant y_l$ and $p(1/l) \geqslant 1$ we have

$$p(y) \geqslant p(y_l) \geqslant p(\frac{p(y)}{l}) = p(y)p(\frac{1}{l}) \geqslant p(y).$$

So $p(1/l) = 1$ and the required equality has been proved.

Thus we should prove that $\partial^- p(y) = A$. Let $l \in A$. First we check that $\langle l, x \rangle^- \leqslant p(x)$ for all $x \in \mathbb{R}^n_+$. Assume on the contrary that there exists $z \in \mathbb{R}^n_+$ such that $\langle l, z \rangle^- = \min_{i \in \mathcal{T}(l)} l_i z_i > p(z)$. Then $l_i z_i > p(z)$ for all $i \in \mathcal{T}(\ell)$, hence $\mathcal{T}(l) \subset \mathcal{T}(z)$ and there exists $\varepsilon > 0$ such that

$$z_i \geqslant \frac{p(z)}{l_i} + \frac{\varepsilon}{l_i} \quad \text{for all} \quad i \in \mathcal{T}(l).$$

Thus $z \geqslant (p(z) + \varepsilon)(1/l)$. Since p is increasing it follows that

$$p(z) \geqslant p((p(z) + \varepsilon)\frac{1}{l}) = (p(z) + \varepsilon)p(\frac{1}{l}) \geqslant p(z) + \varepsilon.$$

We have a contradiction, hence $\langle l, x \rangle^- \leqslant p(x)$ for all $x \in \mathbb{R}^n_+$. Consider now the vector y. Since $\mathcal{T}(l) \subset \mathcal{T}(y)$ and $l \geqslant p(y)/y_l$, it follows that $l_i y_i \geqslant p(y)$ for all $i \in \mathcal{T}(l)$. Thus $\langle l, y \rangle^- \geqslant p(y)$. The inequality $\langle l, y \rangle^- \leqslant p(y)$ has already been proved. Thus $\langle l, y \rangle^- = p(y)$ and therefore $\partial^- p(y) \supset A$.

We now check that the set A is not empty. Indeed let $l = p(y)/y$. Then $\mathcal{T}(l) = \mathcal{T}(y)$. Since $p(y) \neq 0$, the vector $1/l = y/p(y)$ is well-defined and $p(1/l) = 1$. Thus $l \in A$. We have proved that the subdifferential $\partial^- p(y)$ is not empty.

Let us show that $\partial^- p(y) \subset A$ if $p(y) > 0$. Let $l \in \partial^- p(y)$ that is $\langle l, x\rangle^- \leqslant p(x)$ for all x and $\min_{i\in\mathcal{T}(l)} l_i y_i = \langle l, y\rangle^- = p(y)$. Since $p(y) > 0$ it follows that $y_i > 0$ for $i \in \mathcal{T}(l)$, hence $\mathcal{T}(l) \subset \mathcal{T}(y)$. The inequality $l_i y_i \geqslant p(y)$ for all $i \in \mathcal{T}(l)$ shows that $l \geqslant p(y)/y_l$. We also have

$$p(\frac{1}{l}) \geqslant \langle l, \frac{1}{l}\rangle^- = 1$$

Thus $l \in A$. □

COROLLARY 5.1. *If $p(y) > 0$ then $l = \dfrac{p(y)}{y} \in \partial^- p(y)$.*

COROLLARY 5.2. *An IPH function is abstract convex with respect to the set $\mathcal{L}^-$ of functions of the form (6).*

REMARK 5.1. Consider an IPH function p of $n+1$ variables defined on the cone $\mathbb{R}^{n+1}_* = \{(x, \lambda) : x \in \mathbb{R}^n_+, \lambda > 0\} \cup \{0, 0\}$. Clearly Proposition 5.1 and Corollary 5.2 hold also for this function.

A function p is called *strictly increasing at a point* $y \in \mathbb{R}^n_+$ if for each $x \in \mathbb{R}^n_+$ with $x < y$ the inequality $p(x) < p(y)$ holds. It is clear that $p(y) > 0$ for an IPH function p, strictly increasing at a point y.

PROPOSITION 5.2. *If an IPH function p is strictly increasing at a point y then*

$$(l \in \partial^- p(y),\ \mathcal{T}(l) = \mathcal{T}(y)) \implies l = \frac{p(y)}{y}.$$

Proof: It follows from Proposition 5.1 that we have to check that the relations

$$\mathcal{T}(l) = \mathcal{T}(y), \quad l \geqslant \frac{p(y)}{y_l}, \quad p(\frac{1}{l}) = 1 \tag{9}$$

imply $l = p(y)/y$. Take l such that (9) holds. Clearly $y \geqslant p(y)/l$. Assume $y \neq p(y)/l$. Since p is strictly increasing at the point y we have

$$p(y) > p(\frac{p(y)}{l}) = p(y)p(\frac{1}{l}) \quad \text{and} \quad p(y) > 0$$

Thus $p(1/l) < 1$ which is a contradiction. □

Let us give some examples.

EXAMPLE 5.1. Let $p(x) = \max_{i\in I} x_i$ and $\mathbf{1} = (1, 1, \ldots, 1)$. Then

$$\partial^- p(\mathbf{1}) = \{l : l \geqslant \mathbf{1}_l,\ \max_{i\in\mathcal{T}(l)} \frac{1}{l_i} = 1\} = \{l : l \geqslant \mathbf{1}_l,\ \min_{i\in\mathcal{T}(l)} l_i = 1\}.$$

EXAMPLE 5.2. Let $p(x) = \min_{i\in I} x_i$. We have

$$\partial^- p(\mathbf{1}) = \{l : l \geqslant \mathbf{1}_l,\ \min_{i\in\mathcal{T}(l)} \frac{1}{l_i} = 1\} = \{l : l \geqslant \mathbf{1}_l,\ \max_{i\in\mathcal{T}(l)} l_i = 1\}.$$

Thus $\partial^- p(\mathbf{1}) = \{\mathbf{1}^K : K \subset I\}$ where $\mathbf{1}^K$ is a vector such that that its ith coordinate is equal to 1 for $i \in K$ and equal to zero for $i \notin K$. In other words, $\partial^- p(\mathbf{1})$ coincides with the set of all non zero vertices of the n-dimensional cube $\{x : 0 \leqslant x \leqslant \mathbf{1}\}$.

EXAMPLE 5.3. Let $p(x) = (x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$. It is easy to check that the IPH function p is strictly increasing at the point $\mathbf{1}$ and the inequality $\langle l, x\rangle^- \leqslant p(x)$ for all x implies $\mathcal{T}(l) = I$. It follows from Proposition 5.2 that $\partial^- p(\mathbf{1}) = \{\mathbf{1}\}$.

We shall now study the superdifferential

$$\partial^+ p(y) = \{l \in \mathbb{R}^n_+ : \langle l, x\rangle^+ \geqslant p(x) \ \ \forall x \in \mathbb{R}^n_+,\ \langle l, y\rangle^+ = p(y)\}$$

of an IPH function p at a point $y \in \mathbb{R}^n_+$ with respect to the set $\mathcal{L}^+$. Let us give an example which shows that the superdifferential $\partial^+ p(y)$ can be empty for some points $y \in \mathbb{R}^n_+$.

EXAMPLE 5.4. Let p be an IPH function such that

(1) there exists a non-zero element y with the property $p(y) = 0$ (in this case y is a boundary point of the cone $\mathbb{R}^n_+$);
(2) if $x^k = x_{(1)} + x^k_{(2)}$ where $x_{(1)}$ is a fixed vector from $\mathbb{R}^n_{++}$, $\mathcal{T}(x^k_{(2)}) \subset \mathcal{T}(y)$ and $\|x^k_{(2)}\| \to +\infty$ as $k \to \infty$, then $p(x^k) \to +\infty$.

Assume $l \in \partial^+ p(y)$. Then $\langle l, y\rangle^+ = \max_{i\in I} l_i y_i = p(y) = 0$, hence $l_i = 0$ for $i \in \mathcal{T}(y)$. Thus $\mathcal{T}(l) \subset (I \setminus \mathcal{T}(y))$. Let $x \gg 0$ be a vector such that $l_i x_i = 1$ for $i \in \mathcal{T}(l)$ and x_i for $i \notin \mathcal{T}(l)$ be sufficiently large so that $p(x) > 1$. (It follows from 2) that such a vector exists). Since $\langle l, x\rangle^+ = 1$, the inequality $\langle l, x\rangle^+ \geqslant p(x)$ does not hold. Thus $\partial^+ p(y)$ is empty.

Consider in particular a function $p(x) = (x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$. It follows from above that the superdifferential $\partial^+ p(y)$ is empty for each boundary point y of the cone $\mathbb{R}^n_+$.

Nevertheless, if $y \gg 0$ then the superdifferential $\partial^+ p(y)$ is nonempty and it contains interior points of the cone $\mathbb{R}^n_+$.

PROPOSITION 5.3. *Let p be an IPH function and $y \gg 0$. Then $\partial^+ p(y) \cap \mathbb{R}^n_{++}$ is nonempty and*

$$\partial^+ p(y) \cap \mathbb{R}^n_{++} = \{l \gg 0 : l \leqslant \frac{p(y)}{y},\ p(\frac{1}{l}) = 1\} \tag{10}$$

Proof: We will use the same arguments as in the proof of Proposition 5.1. First we show that the set on the right hand side of (10) coincides with the following set B:

$$B = \{l \gg 0 : l \leqslant \frac{p(y)}{y},\ p(\frac{1}{l}) \leqslant 1\}.$$

Indeed, if $l \in B$ then $y \leqslant p(y)/l$, hence

$$p(y) \leqslant p(\frac{p(y)}{l}) = p(y)p(\frac{1}{l}).$$

Since $p(y) > 0$ we have $p(1/l) \geqslant 1$. On the other hand $p(1/l) \leqslant 1$. Thus the required equality has been proved. Let $l \in B$. Since $l \leqslant p(y)/y$ it follows that

$$\max_{i \in \mathcal{T}(y)} l_i y_i = \max_{i \in I} l_i y_i = \langle l, y\rangle^+ \leqslant p(y). \tag{11}$$

We now check that $\langle l, x\rangle^+ \geqslant p(x)$ for all $x \in \mathbb{R}^n_+$. Assume, on the contrary, that there exists z such that $\langle l, z\rangle^+ < p(z)$. Then $l_i z_i < p(z)$ for all $i \in I$, therefore $z < p(z)/l$. There exists $\varepsilon > 0$ such that $(1+\varepsilon)z < p(z)/l$. We have

$$(1+\varepsilon)p(z) = p((1+\varepsilon)z) \leqslant p(\frac{p(z)}{l}) = p(z)p(\frac{1}{l}).$$

Since $p(z) > 0$ it follows that $p(\frac{1}{l}) > 1+\varepsilon$. We have a contradiction which shows that $\langle l, x\rangle^+ \geqslant p(x)$ for all x. In particular, $\langle l, y\rangle^+ \geqslant p(y)$. Combining this inequality with (11) we conclude that $\langle l, y\rangle^+ = p(y)$. Thus $l \in \partial^+ p(y)$. Clearly, the element $l = p(y)/y$ belongs to the set B. Thus the superdifferential $\partial^+ p(y)$ is not empty.

Assume now that an element $l \gg 0$ belongs to $\partial^+ p(y)$. Then $1 = \langle l, 1/l\rangle^+ \geqslant p(1/l)$. It follows from the equality $\langle l, y\rangle^+ = p(y)$ that $l \leqslant p(y)/y$. Thus $l \in B$. □

COROLLARY 5.3. *The restriction of IPH function on the cone* $\mathbb{R}^n_{++}$ *is abstract concave with respect to class* $\mathcal{L}^+$.

6. Subdifferentials and superdifferentials of ISSI functions

In this section we study ISSI functions using results of the previous sections.

Let f be an ISSI function. Consider its positively homogeneous extension $\hat{f}$ defined on the cone $\mathbb{R}^{n+1}_*$ (see (4) for the definition of this cone).

It follows from Theorem 4.1 that $\hat{f}$ is an IPH function on the cone $\mathbb{R}^{n+1}_*$, so (see Corollary 5.2 and Remark 5.1) this function is abstract convex with respect to the set $\hat{\mathcal{L}}^-$ of all functions $\hat{l}$ defined on $\mathbb{R}^{n+1}_+$ by the formula (6). We present a vector $\hat{l} \in \mathbb{R}^{n+1}_+$ in the form $\hat{l} = (l, c)$ where $l \in \mathbb{R}^n_+,\ c \geqslant 0$. Let $x \in \mathbb{R}^n_+$ and $\hat{x} = (x, 1)$.

We have

$$\begin{aligned} f(x) &= \hat{f}(\hat{x}) \\ &= \sup\{\min_{i\in\mathcal{T}(\hat{l})} \hat{l}_i\hat{x}_i : \hat{l} \in \hat{\mathcal{L}}^-, \hat{l} \leqslant \hat{f}\} \\ &= \sup\{\min(\min_{i\in\mathcal{T}(l)} l_i x_i, c) : \hat{l} = (l, c), l \in \mathcal{L}^-, c \geqslant 0, \hat{l} \leqslant f\} \end{aligned} \tag{12}$$

It follows from (12) that f is abstract convex with respect to the set $\mathcal{H}^-$ of functions $h : \mathbb{R}^n_+ \to \mathbb{R}_+$ of the form $h(x) = \min(\langle l, x\rangle^-, c)$ with $l \in \mathbb{R}^n_+$ and $c \geqslant 0$.

Let f be an ISSI function and $y \in \mathbb{R}^n_+$. Denote by $\bar{\partial}^- f(y)$ the subdifferential of the function f at the point y with respect to the set $\mathcal{H}^-$. If $f(y) = 0$ then this subdifferential is not empty, it contains the function $h = 0$ which belongs to $\mathcal{H}$. Assume now that $f(y) > 0$. Then $\hat{f}(\hat{y}) = f(y) > 0$ and therefore (see Corollary 5.1)

$$\frac{\hat{f}(\hat{y})}{\hat{y}} = \frac{f(y)}{(y, 1)} \in \partial^- \hat{f}(y).$$

It follows from this inclusion that

$$f(y) = \hat{f}(\hat{y}) = \min(\min_{i\mathcal{T}(y)} \frac{f(y)}{y_i}, \frac{f(y)}{1}) = \min(\langle \frac{f(y)}{y}, y\rangle^-,\ f(y)) \tag{13}$$

and for each $x \in \mathbb{R}^n_+$

$$\min(\langle \frac{f(y)}{y}, x\rangle^-, f(y)) \leqslant \hat{f}(x, 1) = f(x). \tag{14}$$

Let

$$h(x) = \min(\langle \frac{f(y)}{y}, x\rangle^-, f(y)).$$

It is clear that $h \in \mathcal{H}^-$. Formulae (13) and (14) show that f is an element of the subdifferential $\bar{\partial}^- f(x)$ of the function f at the point x with respect to the set $\mathcal{H}^-$. Thus $\bar{\partial}^- f(x)$ is not empty.

We now give a description of the subdifferential $\bar{\partial}^- f(x)$, by assuming that $f(x) > 0$. Applying Proposition 5.1 and Remark 5.1 we can conclude that

$$\partial^- \hat{f}(\hat{x}) = \{\hat{l} = (l, \mu) \in \mathbb{R}^{n+1}_+ : \mathcal{T}(\hat{l}) \subset \mathcal{T}(\hat{x}) : \hat{l} \geqslant \frac{\hat{f}(\hat{x})}{\hat{x}_{\hat{l}}},\ \hat{f}(\frac{1}{\hat{l}}) = 1\} \tag{15}$$

Since

$$\hat{f}(\frac{1}{(l, \mu)}) = \frac{1}{\mu}\hat{f}(\frac{\mu}{l}, 1) = \frac{1}{\mu} f(\frac{\mu}{l})$$

we can present (15) in the following form:

$$\partial^- \hat{f}(\hat{x}) = \{(l, \mu) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : \mathcal{T}(l) \subset \mathcal{T}(x) : l \geqslant \frac{f(x)}{x_l}, \ \mu \geqslant f(x), \ f(\frac{\mu}{l}) = \mu\}. \tag{16}$$

Clearly $\bar{\partial}^- f(x)$ coincides with the set on the right side of (16).

The same arguments show that the restriction of an ISSI function f on the cone $\mathbb{R}^n_{++}$ is abstract concave with respect to the set $\mathcal{H}^+$ of max-type functions h of the form $h(x) = \max(\langle l, x\rangle^+, c)$ with $l \in \mathbb{R}^n_+$ and $c \geqslant 0$. We can easily describe the superdifferential of the function f at a point $y \gg 0$ with respect to the set $\mathcal{H}^+$. In particular the function

$$h(x) = \max(\langle \frac{f(y)}{y}, x\rangle^+, f(y))$$

belongs to the superdifferential.

7. Algorithm

We consider the following optimization problem

$$f(x) \longrightarrow \min \quad \text{subject to} \quad x \in X \tag{17}$$

where f is an ISSI function (in particular, it can be an IPH function), X is a closed convex set from $\mathbb{R}^n_+$.

We propose for the solution of this problem a variant of the Φ-*bundle method*, the conceptual scheme of which can be found in (see [16]). Note that for IPH functions it is possible to apply the so called "cutting angle method" proposed in [2]. However, the method which we consider in this paper, is different, as instead of the minorants of the type $\min_i \ell_i x_i + c$ we use the functions $\min\{\min_i \ell_i x_i, c\}$. The algorithm can be considered as an extension of the cutting plane method [12] for convex optimization to a problem with an ISSI objective function.

ALGORITHM 1. Method of minimizing an ISSI function

Step 0. Let $k := 0$. Choose an arbitrary initial point $x_o \in X$.
Step 1. Calculate an element of the subdifferential $(h_k, c_k) \in \partial_L f(x_k)$. Denote $f_k(x) = \min(\langle h_k, x\rangle^-, c_k)$.
Step 2. Solve the following subproblem

$$\max_{0\leqslant i\leqslant k} f_i(x) \longrightarrow \min \quad \text{subject to} \quad x \in X. \tag{18}$$

Step 3. Let y^* be a solution of the problem (18). Let $k := k+1$, $x_k = y^*$ and go to step 1.

The convergence theorem for this algorithm is the following.

THEOREM 7.1. *Let X be a compact set and the directional derivatives $f_k'(x)$ be uniformly bounded on the set X:*

$$\|f_k'(x)\| = \max_{\|u\|\leqslant 1} |f_k'(x,u)| \leqslant R < +\infty \quad \textit{for all} \quad x \in X, k = 0, 1, \ldots \tag{19}$$

Then each limit point x of the sequence (x_k) is a global minimum point of the problem (17).

Proof: Obviously, $f_k(x)$ are concave functions. As the directional derivatives of them are bounded, the conditions of a theorem of convergence from [2] are satisfied. Applying this theorem, we obtain the convergence result. □

If we have a maximization problem,

$$f(x) \longrightarrow \max \quad \text{subject to} \quad x \in X, \tag{20}$$

where f is an ISSI function and $X \subset \mathbb{R}^n_{++}$, the situation is similar. Instead of an element of the subdifferential we need to take an element of the superdifferential and instead of minimizing the maximum of $f_k(x)$ we need to maximize the minimum of f_k. The same convergence result holds (the proof is analogous).

In general, it is difficult to apply a $\Phi-$bundle method for minimizing an abstract convex function, as there is no explicit formula for its subdifferential. The main advantage of ISSI functions is that it is possible to calculate an element of the abstract subdifferential numerically. This allows to construct a fully implementable algorithm of their minimization.

8. Solution of the subproblem

The crucial part of our method is the solution of the auxiliary subproblem in Step 2. Let us express the subproblem in the following form:

$$\begin{aligned} t &\to \min \\ \min(\langle h_i, x\rangle^-, c_i) \leqslant t \quad \forall i &= \overline{0,k} \\ s.t. \quad x &\in X \end{aligned} \tag{21}$$

As $c_i \geqslant 0$ for all i due to the properties of ISSI functions, the optimal value of t is in the interval $[0, \max_{i=\overline{0,k}} \min(\langle h_i, x_0\rangle^-, c_i)]$. Applying the dichotomy with respect to t, we obtain a sequence of the systems:

$$\begin{aligned} \min(\langle h_i, x\rangle^-, c_i) \leqslant \tilde{t} \quad \forall i &= \overline{0,k}, \\ x &\in X, \end{aligned}$$

where $\tilde{t}$ is some fixed value in the interval $[0, \max_{i=\overline{0,k}} \min(\langle h_i, x_0\rangle^-, c_i)]$.

Then, if for some j we have $c_j \leqslant \tilde{t}$, the corresponding inequality is redundant and it can be eliminated. Vice versa, if $c_j > \tilde{t}$, then the corresponding constant c_j

is redundant and the same constraint can be rewritten without the constant. Thus we obtain the following system for each $\tilde{t}$:

$$\begin{aligned} \langle h_j, x\rangle^- \leqslant \tilde{t} \quad & \forall j \in J, \\ & x \in X, \end{aligned}$$

where $J = \{j \in 0, 1, \dots , k\} : c_j > \tilde{t}\}$.

This system was studied in [2], where an algorithm for finding its solution was proposed (similar to dynamic programming). Applying this algorithm, we can obtain a solution of the subproblem (21).

Another possibility is to reduce the system (21) to a problem of mixed integer linear programming with 0-1 variables (see [3] and the references therein). It is a well known technique which is based on introducing a large positive parameter M. Then each constraint $\langle h_j, x\rangle^- \leqslant \tilde{t}$ can be represented as $n + 1$ linear constraints of the form:

$$\begin{aligned} & h_{j1}x_1 - My_{j1} \leqslant \tilde{t} \\ & h_{j2}x_2 - My_{j2} \leqslant \tilde{t} \\ & \dots\dots\dots \\ & h_{jn}x_n - My_{jn} \leqslant \tilde{t} \\ & \sum_{i=1}^{n} y_{ji} \leqslant M - 1, \end{aligned}$$

where $y_{ji} \in \{0, 1\}$ for all i, j. If M is sufficiently large, this problem is equivalent to the subproblem (21). This technique increases the dimensionality of the problem by n binary variables in each iteration. However, it allows to apply existing software packages for mixed integer linear programming in order to solve the subproblem.

9. Results of numerical experiments

A number of numerical experiments have been carried out for test examples, most of which have multiple local minima which are not global. It has appeared that some practical problems arising in engineering have an ISSI objective function and a convex feasible set. In many cases it is possible to transform the initial problem in order to reduce it to the problem (17).

Let us consider now the following test examples taken from the books [5, 7].

EXAMPLE 1. [7]

$$\begin{aligned}&0.01x_1^2 + x_2^2 \to \min\\&x_1x_2 \geqslant 25\\&x_1^2 + x_2^2 \geqslant 25\\&2 \leqslant x_1 \leqslant 50\\&0 \leqslant x_2 \leqslant 50.\end{aligned}$$

The objective function in this (also, the next) example is not ISSI, but it is a nonnegative quadratic function, so its square root was minimized instead which is an IPH (hence, also an ISSI) function. The optimal solution of this problem is the point $(\sqrt{250}; \sqrt{2.5})$ The best found solution is:

$$x_1 = 15.811402; x_2 = 1.581137.$$

The absolute precision by maximum norm is 0.000014 and the relative precision by the objective function is 0.000003. In order to find this solution, the algorithm required 18 iterations.

EXAMPLE 2. [7]

$$\begin{aligned}&x_1^2 + x_2^2 + x_3^2 \to \min\\&x_1^2 + x_2^2 - 1 \geqslant 0\\&1 \leqslant x_1 \leqslant 10\\&-10 \leqslant x_2 \leqslant 10\\&-10 \leqslant x_3 \leqslant 10.\end{aligned}$$

As the feasible domain for this test problem does not belong to $\mathbb{R}^n_+$, it is necessary to divide it into several parts and minimize the objective function over each of them. The optimal solution of the problem is the point $(1; 0; 0)$. The best found solution is:

$$x_1 = 1.000000; x_2 = 0.000000; x_2 = 0.000000.$$

The absolute precision by maximum norm is 0 and the relative precision by the objective function is 0. In order to find this solution, the algorithm required 20 iterations. Note that 12 iterations were necessary in order to find the optimal solution within the precision 0.001.

In the next three examples the objective function is a sum of an ISSI and a linear functions. It has the following form:

$$f(x) = g(x) + [c, x],$$

where g is an ISSI function and $c \in \mathbb{R}^n$. In this case we introduced a new variable $v = [c, x]$ for the linear part, thus the objective function becoming ISSI on

$\mathbb{R}^n_+$. However, as in the optimal solution v could be negative, it was necessary to adjust slightly the subproblem, including v inside min-type functions. Namely, we generated subproblems of the following type:

$$\begin{aligned} & t \to \min \\ & \min(\langle h_i, x\rangle^- + v, c_i + v) \leqslant t \quad \forall i = \overline{0, k} \\ & s.t. \quad x \in X \end{aligned} \tag{22}$$

which can be reduced to mixed integer programming problems using the same technique.

It is important to note that the algorithm allows to check optimality of the best found point. If two subsequent iterates coincide, this means that they are a global minimum point of the objective function (see [2]). The optimality has been confirmed for examples 3 and 4 in this way.

EXAMPLE 3. [5]

$$\begin{aligned} & x_1^{0.6} + x_2^{0.6} - 6x_1 - 4u_1 + 3u_2 \to \min \\ & x_2 - 3x_1 - 3u_1 = 0 \\ & x_1 + 2u_1 \leqslant 4 \\ & x_2 + 2u_2 \leqslant 4 \\ & x_1 \leqslant 3 \\ & u_2 \leqslant 1 \end{aligned}$$

The optimal solution of this problem is the point $(4/3; 4; 0; 0)$ The best found solution is:

$$x_1 = 1.333333; x_2 = 3.999999;$$

$$u_1 = 0.000000; u_2 = 0.000000.$$

The absolute precision by maximum norm is 0.000001 and the relative precision by the objective function is $3 \cdot 10^{-7}$. In order to find this solution, the algorithm required 12 iterations.

EXAMPLE 4. [5]

$$\begin{aligned} & x_1^{0.6} + 2x_2^{0.6} + 2u_1 - 2x_2 - u_2 \to \min \\ & x_2 - 3x_1 - 3 = 0 \\ & x_1 + 2u_1 \leqslant 4 \\ & x_2 + u_2 \leqslant 4 \\ & x_1 \leqslant 3 \\ & u_2 \leqslant 2 \end{aligned}$$

Unfortunately, there is a misprint in the source of the test problem. The algorithm gave the following best solution:

$$x_1 = 0.000000; x_2 = 2.999999;$$

$$u_1 = 0.000000; u_2 = 0.999999.$$

The absolute precision by maximum norm is 0.000001 and the relative precision by the objective function is $2 \cdot 10^{-7}$. In order to find this solution, the algorithm required 11 iterations.

EXAMPLE 5. [5]

$$x_1^{0.6} + x_2^{0.6} + x_3^{0.4} + 2u_1 + 5u_2 - 4x_3 - u_3 \to \min$$
$$x_2 - 3x_1 - 3u_1 = 0$$
$$x_3 - 2x_2 - 2u_2 = 0$$
$$4u_1 - u_3 = 0$$
$$x_1 + 2u_1 \leqslant 4$$
$$x_2 + u_2 \leqslant 4$$
$$x_3 + u_3 \leqslant 6$$
$$x_1 \leqslant 3$$
$$u_2 \leqslant 2$$
$$x_3 \leqslant 4$$

The optimal solution is the point $(2/3; 2; 4; 0; 0; 0)$. The best found solution is the following point:

$$x_1 = 0.666666; x_2 = 2.000000; x_3 = 3.999999$$

$$u_1 = 0.000000; u_2 = 0.000000; u_3 = 0.000000.$$

The absolute precision by maximum norm is 0.000001 and the relative precision by the objective function is 10^{-7}. In order to find this solution, the algorithm required 27 iterations.

EXAMPLE 6. [7]

$$9x_1^2 + x_2^2 + 9x_3^2 \to \min$$
$$x_1x_2 \geqslant 1$$
$$-10 \leqslant x_2 \leqslant 10$$
$$1 \leqslant x_2 \leqslant 10$$
$$-10 \leqslant x_3 \leqslant 1.$$

Note that in this example the variables are not in $\mathbb{R}^n_+$, thus it is necessary to subdivide the feasible region in order to get the optimal point. The objective function is not ISSI, but it is nonnegative, so its square root was minimized instead of it (which is an IPH function). This has been a difficult test for the algorithm. The optimal solution of this problem is the point $(1/\sqrt{3}; \sqrt{3}; 0)$. The best found solution is:

$$x_1 = 0.577328;\ x_2 = 1.732118;\ x_2 = 0.000000$$

The absolute precision by maximum norm is 0.000067 and the relative precision by the objective function is $2 \cdot 10^{-7}$. In order to find this solution, the algorithm required 72 iterations. In order to find the optimal solution within the precision 0.01 only 31 iterations were required. Probably, for problems of similar type it is preferable to use hybrid methods, combining the global algorithm with local search (in case of ICAR functions it allowed us to increase significantly the dimensionality of solvable problems, see [2]).

EXAMPLE 7. [7]

$$\begin{aligned}
&2 - x_1x_2x_3 \to \max \\
&x_1 + 2x_2 + 2x_3 - x_4 = 0 \\
&0 \leqslant x_i \leqslant 1 \;\; i = \overline{1,3}, \\
&0 \leqslant x_4 \leqslant 2.
\end{aligned}$$

The optimal solution of this problem is the point $(2/3; 1/3; 1/3; 2)$. The best found solution is:

$$x_1 = 0.667039;\ x_2 = 0.333240;$$

$$x_3 = 0.333240;\ x_4 = 2.000000.$$

As the product of variables is not an ISSI function, we minimized instead of it the cubic root of the objective function which is an IPH function. The absolute precision by maximum norm is 0.000372 and the relative precision by the objective function is $7 \cdot 10^{-8}$. In order to find this solution, the algorithm required 57 iterations.

Except these, a number of numerical experiments have been carried out with Cobb-Douglas type objective functions which are very important in mathematical economics.

COBB-DOUGLAS TYPE FUNCTION

$$f(x) = \prod_{i=1}^{n} x_i^{\alpha_i},\ \alpha_i > 0.$$

Note that if $\sum_{i=1}^{m} a_i < 1$ then we have an ISSI function, if $\sum_{i=1}^{s} \alpha_i \geqslant 1$ we have an increasing convex-along-rays (ICAR(X)) function [20]. A method for minimizing ICAR(X) functions (a cutting angle method) was considered in [2] and it is very efficient for small dimensions.

We give now the comparison between the performance of the algorithm for ISSI functions and the cutting angle method. Note that for ISSI objective functions and linear constraints the subproblem was solved by the software package CPLEX MIP 4.0 (after its reduction to a problem of mixed integer programming). The precision of solution for Algorithm 1 is clearly better, but the method of solving the subproblem which is described in [2], is somewhat faster by time.

The constraint function g was chosen either as a convex differentiable function, or as a maximum of linear functions, or as a maximum of convex quadratic and linear functions. Except this, problems with a constraint, given by a decreasing convex or concave function, were considered.

The code has been written in Borland C++ for Windows 95 and has been also implemented for the AIX 3.2 operating system on IBM RS 6000. The initial point was chosen by convex optimization.

In the Table 1 - Table 2 (Appendix) we present the comparison of computational results for Algorithm 1 and the cutting angle method for ten examples with Cobb-Douglas type objective functions. The space dimension in these examples was taken as equal to four. In the tables the precision, obtained after 50 iterations, is given. In the third column we give the precision by norm $\|x_r - x^*\|$ where x^* is the exact optimal solution which is known and x_r is the best found feasible point. In the last column we give the relative precision by objective which is equal to $(f(x_r) - f(x^*))/(f(x_0) - f(x^*))$. In all cases the algorithm found a global minimum within the precision $\varepsilon = 0.0001$.

The subproblem in Step 2 is of essentially combinatorial nature. This means that, in general, its complexity grows exponentially with respect to the number of variables. The present techniques of solving the subproblem allow us to solve problems with 8-10 variables. Note, however, that if the number of variables is fixed, the complexity with respect to the number of min-type functions is polynomial.

The experiments show that the algorithm considered in this paper can outperform the cutting angle method in many cases, probably due to the fact that some constraints in the subproblem are redundant and this reduces its dimensionality. It is important that the algorithm is applicable not only to minimization, but also to maximization problems with increasing positively homogeneous functions.

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Appendix

Table 1. Results for Algorithm 1

Number of problem	Number of iterations	Precision by norm	Relative precision by objective
1	50	0.000007	0.000012
2	50	0.000012	0.000024
3	50	0.000022	0.000109
4	50	0.000004	0.000006
5	50	0 000005	0.000001
6	50	0 000033	0.000007
7	50	0.000055	0.000028
8	50	0 000003	0.000009
9	50	0.000044	0.000086
10	50	0.000008	0.000010

Table 2. Results for cutting angle method

Number of problem	Number of iterations	Precision by norm	Relative precision by objective
1	50	0.007460	0.003552
2	50	0.005095	0.000606
3	50	0.002252	0.000393
4	50	0.002821	0.000799
5	50	0.010630	0.001746
6	50	0.001661	0.000071
7	50	0.000776	0.001035
8	50	0.001048	0.000342
9	50	0.000690	0.001067
10	50	0.000347	0.000960

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